

THE SMALLEST ENCLOSING BALL PROBLEM AND THE SMALLEST INTERSECTING BALL PROBLEM: EXISTENCE AND UNIQUENESS OF SOLUTIONS

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Abstract: In this paper we study the following problems: given a finite number of nonempty closed subsets of a normed space, find a ball with the smallest radius that encloses all of the sets, and find a ball with the smallest radius that intersects all of the sets. These problems can be viewed as generalized versions of the *smallest enclosing circle problem* introduced in the 19th century by Sylvester [12] which asks for the circle of smallest radius enclosing a given set of finite points in the plane. We will focus on the sufficient conditions for the existence and uniqueness of an optimal solution for each problem, while the study of optimality conditions and numerical implementation will be addressed in our next projects.

Key words. smallest enclosing ball problem, smallest intersecting ball problem

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1 Introduction and Problem Formulation

Let X be a normed space, and let $F \subset X$ be a closed, bounded, convex set which contains the origin as an interior point. Given $x \in X$ and $r \geq 0$, we define

$$D_F(x; r) = x + rF$$

to be a closed bounded convex set centered about the point x with radius r .

The first problem we propose and study in this paper is called the *smallest enclosing ball problem* and is stated as follows: given a nonempty closed constraint set $\Omega \subset X$ and a finite number of nonempty closed bounded subsets $\Omega_i \subset X$ for $i = 1, \dots, n$, find a point $\bar{x} \in \Omega$ and the smallest radius $r \geq 0$ such that the set $D_F(\bar{x}; r)$ contains all of the sets, that is, $\Omega_i \subset D_F(\bar{x}; r)$ for $i = 1, \dots, n$.

The second problem under consideration is called the *smallest intersecting ball problem* and is stated as follows: given a nonempty closed constraint set $\Omega \subset X$ and a finite number of nonempty closed subsets $\Omega_i \subset X$ for $i = 1, \dots, n$, find a point $\bar{x} \in \Omega$ with smallest radius $r \geq 0$ such that the set $D_F(\bar{x}; r)$ intersects all of the sets Ω_i .

When F is the closed unit ball in the Euclidean plane and the *target sets* Ω_i , $i = 1, \dots, n$, are singletons and the constraint set Ω is the whole plane, then both problems reduce to the *classical smallest enclosing circle problem* introduced by the English mathematician James Joseph Sylvester (1814–1897) which asks for the smallest circle that covers a finite number of points on the plane. After more than a century, the smallest enclosing circle problem remains active; see [1, 3, 11, 13] and the references therein. The reader is referred to our recent paper [10] for a comprehensive study of the convex version of the smallest intersecting ball problem. The results presented in

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this paper and its continuation further our idea of using variational/nonsmooth analysis and optimization to shed new light on classical geometry problems.

Given a nonempty closed bounded set $Q \subset X$, we define the *maximal time function* of $x \in X$ given Q generated by F as follows

$$C_F(x; Q) = \inf\{t \geq 0 : Q \subset x + tF\}. \quad (1.1)$$

When F is the closed unit ball of X , the maximal time function (1.1) reduces to the corresponding *farthest distance function*

$$M(x; Q) = \sup\{\|x - \omega\| : \omega \in Q\}.$$

General and generalized differentiation properties of farthest distance functions can be found, for instance, in [2, 4, 14].

The *minimal time function* counterpart is defined below as

$$T_F(x; Q) = \inf\{t \geq 0 : (x + tF) \cap Q \neq \emptyset\}, \quad (1.2)$$

where Q needs not necessarily be bounded. The minimal time function (1.2) is more well-known in the literature; see, e.g. [6] and the references therein. It becomes the familiar distance function

$$d(x; Q) = \inf\{\|x - q\| : q \in Q\}$$

when F is the closed unit ball of X .

In this paper, we will show that under natural assumptions, the smallest enclosing ball problem can be modeled in terms of an optimization problem as follows:

$$\text{minimize } C(x) \text{ subject to } x \in \Omega, \quad (1.3)$$

where

$$C(x) = \max\{C_F(x; \Omega_i) : i = 1, \dots, n\}. \quad (1.4)$$

Similarly, the smallest intersecting ball problem can also be converted to the following optimization problem:

$$\text{minimize } T(x) \text{ subject to } x \in \Omega, \quad (1.5)$$

where

$$T(x) = \max\{T_F(x; \Omega_i) : i = 1, \dots, n\}. \quad (1.6)$$

The unconstrained versions of these problems are obtained when $\Omega = X$.

Our goal in this paper and its continuation is to initiate comprehensive studies of the smallest enclosing ball problem and the smallest intersecting ball problem using modern tools of variational analysis and optimization. The main focus of the paper is on sufficient conditions that guarantee the existence and uniqueness of an optimal solution for each problem. In Section 2, we provide important properties of the maximal time function (1.1) and then pay attention to optimality conditions of the smallest enclosing ball problem. Section 3 is devoted to the smallest intersecting ball problem counterpart. Along the way, we point out major differences between these two problems and provide examples to support the need for the assumptions. For instance, in finite dimensional Euclidean space, the smallest enclosing ball problem usually has a unique optimal solution even if the target sets are nonconvex, while strict convexity assumptions must be made to guarantee a unique solution for the smallest intersecting ball problem.

2 The Smallest Enclosing Ball Problem

In this section we initially describe some properties of the maximal time function (1.1) and then prove existence and uniqueness of the solution to the smallest enclosing ball problem in Theorems 2.4 and 2.6, respectively. Finally, we provide some examples that illustrate the need for the assumptions to guarantee uniqueness of the solution.

Throughout this section we make the following standing assumptions unless otherwise noted:

X is a normed space; $F \subset X$ is a closed, bounded, convex set that contains the origin as an interior point; the target sets Ω_i , $i = 1, \dots, n$, are nonempty closed bounded subsets of X ; and the constrained set Ω is a nonempty closed subset of X .

Let us start with some important properties of the maximal time function (1.1). Recall that the Minkowski function generated by F is given by

$$\rho_F(x) = \inf\{t \geq 0 : x \in tF\}. \quad (2.1)$$

The following proposition allows us to represent the maximal time function (1.1) in terms of the Minkowski function (2.1).

Proposition 2.1. *Suppose that Q is a nonempty bounded set of X . Then the maximal time function (1.1) has the following representation:*

$$C_F(x; Q) = \sup\{\rho_F(\omega - x) : \omega \in Q\}.$$

Moreover, if F is the closed unit ball of X , then

$$C_F(x; Q) = \sup\{\|x - \omega\| : \omega \in Q\}.$$

Proof: Define

$$\tilde{C}_F(x; Q) = \sup\{\rho_F(\omega - x) : \omega \in Q\}.$$

Fix any $t \geq 0$ such that $Q \subset x + tF$. Then for every $\omega \in Q$ one has

$$\omega - x \in tF.$$

Thus $\rho_F(\omega - x) \leq t$. It follows that

$$\tilde{C}_F(x; Q) \leq C_F(x; Q).$$

Given any $\varepsilon > 0$, one has

$$\sup\{\rho_F(\omega - x) \mid \omega \in Q\} < \tilde{C}_F(x; Q) + \varepsilon.$$

Thus

$$\rho_F(\omega - x) < \tilde{C}_F(x; Q) + \varepsilon \text{ for every } \omega \in Q.$$

By the definition of the Minkowski function, there exists $t \geq 0$ with $t < \tilde{C}_F(x; Q) + \varepsilon$ and

$$\omega - x \in tF.$$

Since F is convex and $0 \in F$, one has

$$\omega - x \in (\tilde{C}_F(x; Q) + \varepsilon)F.$$

This implies

$$Q \subset x + (\tilde{C}_F(x; Q) + \varepsilon)F,$$

and hence $C_F(x; Q) \leq \tilde{C}_F(x; Q) + \varepsilon$. We finally have

$$C_F(x; Q) \leq \tilde{C}_F(x; Q)$$

because ε is arbitrarily chosen. Thus $C_F(x; Q) = \tilde{C}_F(x; Q)$, and the first representation has been proven. In the case where F is the closed unit ball of X , one has $\rho_F(x) = \|x\|$. Therefore, the second representation becomes straightforward. \triangle

For a point $x \in X$, the *farthest projection* from x to a nonempty, closed, bounded set Q with respect to F is defined by

$$\mathcal{P}_F(x; Q) = \{\omega \in Q : \rho_F(\omega - x) = C_F(x; Q)\}.$$

This set is obviously nonempty when Q is a compact subset of X . The following proposition provides a geometric way to realize the set.

Proposition 2.2. *Let Q be a nonempty, closed, bounded, subset of X . Then*

$$\mathcal{P}_F(x; Q) = Q \cap (x + C_F(x; Q)\text{bd}(F)),$$

where $\text{bd}(F)$ stands for the boundary of F .

Proof: Fix any $\omega \in \mathcal{P}_F(x; Q)$. Then $\omega \in Q$ and

$$\rho_F(\omega - x) = C_F(x; Q).$$

This implies $\omega - x \in C_F(x; Q)F$. When $C_F(x; Q) = 0$, it is obvious that

$$\omega \in Q \cap (x + C_F(x; Q)\text{bd}(F)) = \{x\}.$$

Suppose $C_F(x; Q) > 0$. Then

$$\rho_F\left(\frac{\omega - x}{C_F(x; Q)}\right) = 1.$$

By the well-known property of the Minkowski function, this equality implies that $\frac{\omega - x}{C_F(x; Q)} \in \text{bd}(F)$, and hence $\omega \in x + C_F(x; Q)\text{bd}(F)$. We have shown that

$$\mathcal{P}_F(x; Q) \subset Q \cap (x + C_F(x; Q)\text{bd}(F)).$$

The opposite inclusion can also be proved similarly. \triangle

Recall that a function $\phi : X \rightarrow \mathbb{R}$ is convex on a convex set Ω if for every $x, y \in \Omega$ and $t \in (0, 1)$, one has

$$\phi(tx + (1 - t)y) \leq t\phi(x) + (1 - t)\phi(y).$$

If this inequality becomes strict for every $x, y \in \Omega$ with $x \neq y$ and for every $t \in (0, 1)$, the function ϕ is called *strictly convex*.

It is clear that if a function ϕ is strictly convex on a convex set Ω , then the problem

$$\text{minimize } \phi(x) \text{ subject to } x \in \Omega$$

cannot have more than one solution.

We also say that a set Ω is strictly convex if for any $x, y \in \Omega$ with $x \neq y$ and for any $t \in (0, 1)$ we have

$$tx + (1 - t)y \in \text{int } \Omega.$$

Proposition 2.3. *Let Q be a nonempty bounded subset of X . Then the maximal time function (1.1) is a finite convex Lipschitz function.*

Proof: For each $\omega \in Q$, the function

$$g_\omega(x) = \rho_F(\omega - x)$$

is a convex Lipschitz function since the Minkowski function $\rho_F(x)$ given in (2.1) is always Lipschitz continuous with Lipschitz constant ℓ . Since $\rho_F(0) = 0$, one has $\rho_F(x) \leq \ell\|x\|$ for all $x \in X$. It follows from Proposition (2.1) that the function

$$C_F(x; \Omega) = \sup_{\omega \in Q} g_\omega(x)$$

is also a finite convex Lipschitz function under the boundedness assumption imposed on Q . \triangle

Theorem 2.4. *Suppose one of the following holds:*

- (i) *The constraint Ω is a nonempty compact set.*
- (ii) *X is a reflexive Banach space and the constraint set Ω is weakly closed.*

Then the smallest enclosing ball problem has a solution. That means there exist $\bar{x} \in \Omega$ and $r \geq 0$ such that

$$\Omega_i \subset \bar{x} + rF \text{ for } i = 1, \dots, n,$$

and for any $x \in \Omega$ and $t \geq 0$ such that $\Omega_i \subset x + tF$ for $i = 1, \dots, n$, one has $r \leq t$.

Proof: Define

$$\mathcal{E} = \{t \geq 0 : \text{there exists } x \in \Omega \text{ with } \Omega_i \subset x + tF \text{ for all } i = 1, \dots, n\}.$$

Clearly, $\mathcal{E} \neq \emptyset$ and is bounded below. Indeed, fix $\bar{x} \in \Omega$. Since F is convex, $0 \in \text{int}F$, and Ω_i is bounded for every $i = 1, \dots, n$, there exists $\bar{t} > 0$ such that

$$\Omega_i \subset \bar{x} + \bar{t}F \text{ for all } i = 1, \dots, n.$$

Then $\bar{t} \in \mathcal{E} \neq \emptyset$. Define $r = \inf \mathcal{E}$. Let (t_k) be a sequence in \mathcal{E} that converges to r . Let (x_k) be a sequence of Ω such that

$$\Omega_i \subset x_k + t_k F \text{ for all } i = 1, \dots, n.$$

Now we will consider each case given in the assumptions. In the case (i) where Ω is compact, the sequence (x_k) has a subsequence (without relabeling) that converges to some $\bar{x} \in \Omega$ since Ω is closed and in this case

$$\Omega_i \subset \bar{x} + rF \text{ for all } i = 1, \dots, n. \tag{2.2}$$

Let us now consider case (ii) where X is a reflexive Banach space and Ω is weakly closed. It is clear that the sequence (x_k) is bounded, and hence it has a subsequence (without relabeling) that converges weakly to $\bar{x} \in \Omega$. Then (2.2) also holds true.

Now let $x \in \Omega$ and $t \geq 0$ satisfy $\Omega_i \subset x + tF$ for $i = 1, \dots, n$. Then $t \in \mathcal{E}$, and hence $r \leq t$. Thus the smallest enclosing ball problem has a solution, and the proof is now complete. \triangle

Proposition 2.5. *An element $\bar{x} \in \Omega$ is a solution of the optimization problem (1.3) with $r = C(\bar{x})$ if and only if \bar{x} is a solution of the smallest enclosing ball problem with smallest radius r .*

Proof: Suppose that $\bar{x} \in \Omega$ is a solution of the optimization problem (1.3) with $r = C(\bar{x})$. Then

$$C_F(\bar{x}; \Omega_i) \leq r \text{ for all } i = 1, \dots, n.$$

Thus $\Omega_i \subset \bar{x} + rF$ for all $i = 1, \dots, n$. Let $x \in \Omega$ and $t \geq 0$ satisfy

$$\Omega_i \subset x + tF \text{ for all } i = 1, \dots, n.$$

Define $r' = C(x)$. Then $r \leq r'$. Moreover, it is also clear that $r' \leq t$. Thus $r \leq t$, and \bar{x} is a solution of the smallest enclosing ball problem with smallest radius r .

Now suppose that $\bar{x} \in \Omega$ is a solution of the smallest enclosing ball problem with smallest radius r . We will prove that $C(\bar{x}) = r$ and $C(\bar{x}) \leq C(x)$ for all $x \in \Omega$ to show that \bar{x} is a solution of problem (1.3). Since $\Omega_i \subset \bar{x} + rF$ for all $i = 1, \dots, n$, one has $C(\bar{x}) \leq r$. If $C(\bar{x}) < r$, then choose r' such that $C(\bar{x}) < r' < r$. Then $\Omega_i \subset \bar{x} + r'F$ for all $i = 1, \dots, n$, which results in a contradiction because r is the smallest radius associated with \bar{x} . It follows that $r = C(\bar{x})$. For any $x \in \Omega$, define $t = C(x)$. Then $\Omega_i \subset x + tF$ for all $i = 1, \dots, n$. Therefore, $C(\bar{x}) = r \leq t = C(x)$. The proof is now complete. \triangle

In what follows we will establish sufficient conditions for the uniqueness of the smallest enclosing ball problem.

Theorem 2.6. *Suppose that $X = \mathbb{R}^n$, F is the Euclidean closed unit ball of X , and the constraint Ω is a nonempty closed convex subset of X . Then problem (1.3) has a unique solution.*

Proof: By Proposition 2.5, in order to solve the smallest enclosing ball problem, we only need to solve problem (1.3). The existence of an optimal solution under the assumptions made has been proven in Theorem 2.4. Notice that \bar{x} is a solution of the optimization problem (1.3) if and only if it is a solution of the following problem

$$\text{minimize } C^2(x) \text{ subject to } x \in \Omega$$

where $C^2(x) = \max\{[C_F(x; \Omega_i)]^2 : i = 1, \dots, n\}$.

Since the maximum of a finite number of strictly convex functions on Ω is a strictly convex function on this set, the proof reduces to showing that each function

$$c_i(x) = [C_F(x; \Omega_i)]^2 = \sup_{\omega \in \Omega_i} \|x - \omega\|^2 \text{ for } i = 1, \dots, n$$

is strictly convex, where the definition of $C_F(x; \Omega_i)$ arises from Proposition (2.1). It is obvious that the square norm function is strictly convex on X . Fix $x, y \in \Omega$ with $x \neq y$ and $t \in (0, 1)$. Denote $x_t = tx + (1 - t)y \in \Omega$. Then there exists $\omega_{t_i} \in \Omega_i$ for each i such that

$$\begin{aligned} c_i(x_t) &= \|x_t - \omega_{t_i}\|^2 = \|tx + (1 - t)y - \omega_{t_i}\|^2 \\ &= \|t(x - \omega_{t_i}) + (1 - t)(y - \omega_{t_i})\|^2 \\ &< t\|x - \omega_{t_i}\|^2 + (1 - t)\|y - \omega_{t_i}\|^2 \\ &\leq tc_i(x) + (1 - t)c_i(y). \end{aligned}$$

Therefore, each function $c_i(\cdot)$ is strictly convex on Ω , and thus $C^2(x)$ is strictly convex and a unique solution exists. \triangle

The following examples show that the assumptions made in Theorem 2.6 are essential.

Example 2.7. Let $X = \mathbb{R}^2$ and let F be the Euclidean closed unit ball of X . Consider the smallest enclosing ball problem with the target set $\Omega_1 = \{(0, 0)\}$ and the constraint set

$$\Omega = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}.$$

The constraint set is nonconvex and in this case any point $x \in \Omega$ is a solution of the smallest enclosing ball problem.

Example 2.8. Let $X = \mathbb{R}^2$ with unconstrained set $\Omega = \mathbb{R}^2$. Let $F = [-1, 1] \times [-1, 1]$. Define

$$\Omega_1 = \{(0, 1)\} \text{ and } \Omega_2 = \{(0, -1)\}.$$

Then Ω_1 and Ω_2 are both convex. However, the unconstrained smallest enclosing ball problem with target sets Ω_1 and Ω_2 has infinitely many solutions. In fact, any point of the set

$$L = \{(x_1, x_2) \in X : x_1 \in [-1, 1], x_2 = 0\}$$

is a solution of the problem.

3 The Smallest Intersecting Ball Problem

Throughout this section we make the following standing assumptions unless otherwise stated:

X is a normed space; F is a closed, bounded, convex set that contains the origin as an interior point; the constraint set Ω and the target sets Ω_i , $i = 1, \dots, n$, are nonempty closed subsets of X .

Theorem 3.1. *Assume that one of the following statements holds:*

- (i) *X is finite dimensional, and one of the sets among Ω_i , $i = 1, \dots, n$, and Ω is bounded.*
- (ii) *X is a reflexive Banach space, all of the sets Ω_i , $i = 1, \dots, n$, and Ω are weakly closed, and at least one of them is bounded.*

Then the smallest intersecting ball problem has a solution. In this case there exists $r \geq 0$ and $\bar{x} \in \Omega$ such that

$$(\bar{x} + rF) \cap \Omega_i \neq \emptyset \text{ for all } i = 1, \dots, n,$$

and for any $x \in \Omega$ and $t \geq 0$ with $(x + tF) \cap \Omega_i \neq \emptyset$ for all $i = 1, \dots, n$, one has $r \leq t$.

Proof: Consider the following set

$$\mathcal{I} = \{t \geq 0 : \text{there exists } x \in \Omega \text{ with } (x + tF) \cap \Omega_i \neq \emptyset \text{ for all } i = 1, \dots, n\}.$$

Then \mathcal{I} is nonempty since $0 \in \text{int } F$. Moreover, \mathcal{I} is obviously bounded below. Let

$$r = \inf \mathcal{I} \in [0, \infty).$$

Then there exists a sequence $(t_k) \subset \mathcal{I}$ that converges to r . Let $(x_k) \subset \Omega$ satisfy

$$(x_k + t_k F) \cap \Omega_i \neq \emptyset \text{ for all } i = 1, \dots, n.$$

Then there exist $f_{k,i} \in F$ and $\omega_{k,i} \in \Omega_i$, $k \in \mathbb{N}$, $i = 1, \dots, n$, such that

$$x_k + t_k f_{k,i} = \omega_{k,i} \text{ for all } k \text{ for all } i = 1, \dots, n.$$

Let us focus on case (i). We will first show that (x_k) is bounded under the assumptions made. Without loss of generality, suppose that Ω_1 is bounded. One has

$$x_k + t_k f_{k,1} = \omega_{k,1} \text{ for all } k.$$

Thus

$$\|x_k\| \leq t_k \|f_{k,1}\| + \|\omega_{k,1}\|.$$

Since both F and Ω_1 are bounded, (x_k) is a bounded sequence; thus there exists a convergent subsequence (x_k) (without relabeling). Since F is closed and bounded, and Ω is closed, we can assume that $f_{k,i} \rightarrow f_i \in F$ for $i = 1, \dots, n$ and $x_k \rightarrow \bar{x} \in \Omega$. For $i = 1, \dots, n$, one also has

$$x_k + t_k f_{k,i} = \omega_{k,i} \rightarrow \bar{x} + r f_i \text{ as } k \rightarrow \infty,$$

and $\bar{x} + r f_i \in \Omega_i$ since each Ω_i is a closed set. Moreover, $\bar{x} + r f_i \in \bar{x} + r F$ for each $i = 1, \dots, n$. Thus $(\bar{x} + r F) \cap \Omega_i \neq \emptyset$. For any $x \in \Omega$ and $t \geq 0$ with $(x + t F) \cap \Omega_i \neq \emptyset$ for all $i = 1, \dots, n$, one has $t \in \mathcal{I}$. Thus $r \leq t$.

The proof of the result under case (ii) is similar where the weak convergence of $(x_k), (f_{k,i})$ and $(w_{k,i})$ for $i = 1, \dots, n$ are taken into account. The proof is now complete. \triangle

Lemma 3.2. *Suppose one of the following:*

- (i) X is finite dimensional, and Q is a closed subset of X ;
- (ii) X is reflexive, and Q is a nonempty, weakly, closed subset of X .

Let

$$\bar{t} = T_F(\bar{x}; Q) = \inf\{t \geq 0 : (\bar{x} + t F) \cap Q \neq \emptyset\}.$$

Then $\bar{t} \geq 0$ and $(\bar{x} + \bar{t} F) \cap Q \neq \emptyset$.

Proof: Assume case (i) where X is finite dimensional, and let $\bar{x} \in X$. It is clear that $T_F(\bar{x}; Q)$ is a finite number. Let (t_k) be a sequence of nonnegative integers that converges to $\bar{t} \geq 0$, where

$$(\bar{x} + t_k F) \cap Q \neq \emptyset \text{ for all } k.$$

Let $f_k \in F$ and $q_k \in Q$ satisfy that $\bar{x} + t_k f_k = q_k$. Since F is closed and bounded, we can assume without loss of generality that $f_k \rightarrow f \in F$. Then

$$\bar{x} + t_k f_k = q_k \rightarrow \bar{x} + \bar{t} f \in Q.$$

Therefore, $(\bar{x} + \bar{t} F) \cap Q \neq \emptyset$.

Similarly, we can prove case (ii) where X is reflexive and Q is a nonempty, weakly, closed subset of X . \triangle

Proposition 3.3. *Suppose one of the following:*

- (i) X is finite dimensional;
- (ii) X is reflexive and Ω and $\Omega_i, i = 1, \dots, n$, are nonempty, weakly, closed subsets of X .

Then $\bar{x} \in \Omega$ is a solution of the optimization problem (1.5) with $r = T(\bar{x})$ if and only if $\bar{x} \in \Omega$ is a solution of the smallest intersecting ball problem with smallest radius r .

Proof: Suppose that $\bar{x} \in \Omega$ is a solution of the optimization problem (1.5) and $r = T(\bar{x})$. Let

$$t_i = T_F(\bar{x}; \Omega_i) \text{ for } i = 1, \dots, n.$$

This implies $t_i \leq r$ for all $i = 1, \dots, n$. By Lemma (3.2),

$$(\bar{x} + t_i F) \cap \Omega_i \neq \emptyset \text{ for all } i = 1, \dots, n.$$

Thus $(\bar{x} + rF) \cap \Omega_i \neq \emptyset$, which follows from the fact that $t_i F \subset rF$ for all $i = 1, \dots, n$ under the assumptions that F is convex and $0 \in F$.

Now suppose that $x \in \Omega$ and $t \geq 0$ satisfy

$$(x + tF) \cap \Omega_i \neq \emptyset \text{ for all } i = 1, \dots, n.$$

Then $T_F(x; \Omega_i) \leq t$ for all $i = 1, \dots, n$. This implies $r = T(\bar{x}) \leq T(x) \leq t$. Thus \bar{x} is a solution of the smallest intersecting ball problem with radius r .

Conversely, suppose that $\bar{x} \in \Omega$ is a solution of the smallest intersecting ball problem with smallest radius $r \geq 0$. We will prove that \bar{x} is a solution of the optimization problem (1.5) and $r = T(\bar{x})$. One has that

$$(\bar{x} + rF) \cap \Omega_i \neq \emptyset \text{ for all } i = 1, \dots, n.$$

This implies

$$T(\bar{x}) = \max\{T_F(\bar{x}; \Omega_i) : i = 1, \dots, n\} \leq r.$$

If $T(\bar{x}) < r$, then there exists a real number s such that $T(\bar{x}) < s < r$ and we easily see that $(\bar{x} + sF) \cap \Omega_i \neq \emptyset$ for $i = 1, \dots, n$, which contradicts the minimal property of r . Thus $T(\bar{x}) = r$. Now take any $x \in \Omega$. Define $r' = T(x)$. Then $(x + r'F) \cap \Omega_i \neq \emptyset$ for all $i = 1, \dots, n$. Thus $r \leq r'$ or equivalently $T(\bar{x}) \leq T(x)$. Therefore, \bar{x} is a solution of (1.5). The proof is complete. \triangle

The following theorem provides natural sufficiency conditions guaranteeing the uniqueness of the solution for the smallest intersecting ball problem.

Theorem 3.4. *Let X be a Hilbert space and let F be the closed, unit ball of X . Suppose that Ω is a nonempty, closed, convex set, Ω_i , $i = 1, \dots, n$, are strictly convex, and at least one of the sets among Ω_i , $i = 1, \dots, n$ and Ω is bounded. Suppose further that*

$$\cap_{i=1}^n [\Omega_i \cap \Omega] = \emptyset. \quad (3.1)$$

Then the optimization problem (1.5) has a unique solution.

Proof: The existence of an optimal solution follows from Theorem 3.1.

Since F is the closed unit ball of X , the minimal time function $T_F(\cdot; \Omega_i)$ reduces to the distance function $d(\cdot; \Omega_i)$ for $i = 1, \dots, n$. Moreover, \bar{x} is a solution of the smallest intersecting ball problem if and only if it is a solution to the optimization problem (1.5) by Proposition 3.3.

For $x \in \Omega$, consider the function

$$S(x) = \max\{[d(x; \Omega_i)]^2 : i = 1, \dots, n\}.$$

Then \bar{x} is a solution of problem (1.5) if and only if \bar{x} is a solution of the problem

$$\text{minimize } S(x) \text{ subject to } x \in \Omega.$$

We will prove that S is strictly convex on Ω . Fix $x, y \in \Omega$ with $x \neq y$ and $t \in (0, 1)$. Denote $x_t = tx + (1-t)y \in \Omega$. Let $z \in \Omega$ and define $I(z) := \{i = 1, \dots, n : T(z) = T_F(z; \Omega_i)\}$. Then for $i \in I(x_t)$ we have $T(x_t) = d(x_t; \Omega_i)$. Let $u, v \in \Omega_i$ satisfy that $d(x; \Omega_i) = \|x - u\|$ and $d(y; \Omega_i) = \|y - v\|$. It follows that

$$\begin{aligned}
S(x_t) &= [d(x_t; \Omega_i)]^2 \\
&\leq \|x_t - (tu + (1-t)v)\|^2 \\
&= \|t(x - u) + (1-t)(y - v)\|^2 \quad \text{by definition of } x_t, \\
&= t^2\|x - u\|^2 + 2t(1-t)\langle x - u, y - v \rangle + (1-t)^2\|y - v\|^2 \\
&\leq t^2\|x - u\|^2 + 2t(1-t)\|x - u\| \cdot \|y - v\| + (1-t)^2\|y - v\|^2 \\
&\leq t^2\|x - u\|^2 + t(1-t)(\|x - u\|^2 + \|y - v\|^2) + (1-t)^2\|y - v\|^2 \\
&= t\|x - u\|^2 + (1-t)\|y - v\|^2 \\
&= t[d(x; \Omega_i)]^2 + (1-t)[d(y; \Omega_i)]^2 \\
&\leq tS(x) + (1-t)S(y).
\end{aligned}$$

Thus $S(x)$ is convex on Ω . Moreover,

$$\begin{aligned}
S(x_t) &= [d(x_t; \Omega_i)]^2 \leq [td(x; \Omega_i) + (1-t)d(y; \Omega_i)]^2 \\
&\leq t^2\|x - u\|^2 + t(1-t)(\|x - u\|^2 + \|y - v\|^2) + (1-t)^2\|y - v\|^2 \\
&= t\|x - u\|^2 + (1-t)\|y - v\|^2 \\
&= t[d(x; \Omega_i)]^2 + (1-t)[d(y; \Omega_i)]^2 \\
&\leq tS(x) + (1-t)S(y).
\end{aligned}$$

Now, suppose the equality $S(x_t) = tS(x) + (1-t)S(y)$ holds; we will show this leads to a contradiction. We have

$$\langle x - u, y - v \rangle = \|x - u\| \cdot \|y - v\|, \|x - u\| = \|y - v\| \text{ and } d(tx + (1-t)y; \Omega_i) = td(x; \Omega_i) + (1-t)d(y; \Omega_i),$$

which imply that

$$x - u = y - v \text{ and } d(tx + (1-t)y; \Omega_i) = td(x; \Omega_i) + (1-t)d(y; \Omega_i).$$

This implies $u \neq v$ and $d(tx + (1-t)y; \Omega_i) = d(x; \Omega_i) = d(y; \Omega_i)$. Since $T(x_t) = d(x_t; \Omega_i) > 0$ by (3.1), one has $x_t \notin \Omega_i$. Using the strict convexity of Ω_i , one has $tu + (1-t)v \in \text{int } \Omega_i$. Thus

$$d(x_t; \Omega_i) < \|x_t - [tu + (1-t)v]\| \leq t\|x - u\| + (1-t)\|y - v\| = d(x; \Omega_i).$$

Indeed, let $\delta > 0$ satisfy $B(tu + (1-t)v; \delta) \subset \Omega_i$. Denote $c = tu + (1-t)v$. Then $c + \delta \frac{x_t - c}{\|x_t - c\|} \in \Omega_i$ and

$$\|x_t - \delta \frac{x_t - c}{\|x_t - c\|}\| = \|x_t - c\| - \delta < \|x_t - c\|,$$

which is a contradiction. Therefore, S must be strictly convex, and the problem has a unique solution. The proof is now complete. \triangle

The following two examples illustrate the need for the assumptions in the above theorem that provides the uniqueness of the solution to the smallest intersecting ball problem.

Example 3.5. Let $X = \mathbb{R}^2$ with $\Omega = X$ and let F be the Euclidean closed unit ball in X . Define target sets

$$\Omega_1 = \{(x_1, x_2) \in X : x_2 \geq 1\} \text{ and } \Omega_2 = \{(x_1, x_2) \in X : x_2 \leq -1\}.$$

Note that Ω_1 and Ω_2 are both convex and violate the strict convexity assumptions in Theorem 3.4. In addition, none of the target sets Ω_1 and Ω_2 or Ω is bounded. In this case, the unconstrained smallest intersecting ball problem has infinitely many solutions. In fact, any point of the set

$$K = \{(x_1, x_2) \in X : x_2 = 0\}$$

is a solution of the problem.

Example 3.6. Let $X = \mathbb{R}^2$ with $\Omega = X$. Let $F = [-1, 1] \times [-1, 1]$ be the unit square centered at the origin; thus F violates the assumptions of Theorem 3.4. Define

$$\Omega_1 = \mathcal{B}((0, 2); 1) \text{ and } \Omega_2 = \mathcal{B}((0, -2); 1).$$

Then Ω_1 and Ω_2 are both strictly convex and bounded. However, the unconstrained smallest intersecting ball problem with target sets Ω_1 and Ω_2 has infinitely many solutions. In fact, any point of the set

$$L = \{(x_1, x_2) \in X : x_1 \in [-1, 1], x_2 = 0\}$$

is a solution of the problem.

4 Conclusions

This paper is a part of our project involving *set facility location problems*. The main idea is to consider a much broader situation where singletons in the classical models of facility location problems are replaced by sets. The new extension seems to be interesting for both the theory and applications to various location models, optimal networks, wireless communications, etc. Moreover, it sheds new lights on classical geometry problems.

Our next goal is to study optimality conditions and numerical algorithms for the smallest enclosing ball problem and the smallest intersecting ball problem. Based on the approach we have developed in [7–10], we foresee the potential of success of this future work.

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